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Dark energy generated from a (super-) string effective action with higher-order curvature corrections and a dynamical dilaton

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Abstract. We investigate the possibility of a dark energy universe emerging from an action with higher-order string loop corrections to Einstein gravity in the presence of a massless dilaton. These curvature corrections (up to R^4 order) are different depending upon the type of (super-) string model which is considered. We find in fact that type II, heterotic, and bosonic strings respond differently to dark energy. A dark energy solution is shown to exist in the case of the bosonic string, while the other two theories do not lead to realistic dark energy universes. A detailed analysis of the dynamical stability of the de Sitter solution is presented for the case of a bosonic string. A general prescription for the construction of a de Sitter solution for the low-energy (super-) string effective action is also indicated. Beyond the low-energy (super-) string effective action, when the higher-curvature correction coefficients depend on the dilaton, the reconstruction of the theory from the expansion history of the universe is done with a corresponding prescription for the scalar potentials.

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1 Introduction

The Einstein equations in their original form, with an energy-momentum tensor for standard matter on the right hand side, cannot account for the observed accelerated expansion of our universe. The late-time acceleration of the universe, which is directly supported by supernovae observations, and also indirectly, through observations of the microwave background, of the large-scale structure, of weak lensing, and of baryon oscillations, poses one of the most important challenges to modern cosmology. The standard lore aimed at capturing this important effect is related to the introduction of the energy-momentum tensor of exotic matter with large negative pressure (*dark energy*) in the Einstein equations. The simplest known example of dark energy (for recent reviews, see [1–5]) is provided by the cosmological constant. This does not require any ad hoc assumption for its introduction, as is automatically present in the Einstein equations, by virtue of the Bianchi identities. The field theoretic understanding of Λ is far from being satisfactory. Efforts have recently been made to obtain Λ in the framework of string theory, which leads to a complicated landscape of de Sitter vacua. It is

hard to believe that we happen to live in one of the 10^{100} or more vacua predicted by the theory [6]. One might take the simplified view that, like G, the cosmological constant Λ is a fundamental constant of the classical general theory of relativity and that it should be determined from large-scale observations. It is interesting to remark that the ΛCDM model is consistent with the observations at present. Unfortunately, the non-evolving nature of Λ leads to a non-acceptable fine-tuning problem. We do not know how the present scale of the cosmological constant is related to the Planck or the supersymmetry breaking scale; perhaps, some deep physics is at play here that escapes our present understanding.

The fine-tuning problem, associated with Λ, can be alleviated in scalar field models that do not disturb the thermal history of the universe and can successfully mimic Λ at late times. A variety of scalar fields have been investigated to this end [1, 7, 8]; some of them are motivated by field/string theory and the others are introduced owing to phenomenological considerations. It is quite disappointing that a scalar field description lacks predictive power; given a priori a cosmic evolution, one can always construct a field potential that would give rise to it. These models should, however, not be written off, and should be judged by the generic features that might arise from them. For

instance, the tracker models have remarkable features allowing them to alleviate the fine-tuning and coincidence problems. Present data are insufficient in order to conclude whether or not the dark energy has dynamics; thus, the quest for the metamorphosis of dark energy continues.

The other alternative for getting accelerated expansion is related to modifications of the geometry itself or the left hand side of the Einstein equations. There are several ways of modifying gravity (for a review, see [9]). Higher dimensional (including stringy) effects might lead to large-scale modifications of gravity. Another approach, which is largely motivated by phenomenological considerations, is related to the modification of the form of the gravitational action (like $F(R)$ gravity, etc.). The third intriguing alternative is provided by the higher-order curvature corrections to Einstein gravity due to low-energy (super-) string effective action [10]. The leading order correction in the string expansion parameter α' is given by a Gauss–Bonnet (GB) term, which has several remarkable features and which was proposed as a dark energy model [11]. The next-to-leading corrections, cubic and quartic in the curvature, crucially depend upon the type of string model under consideration. The higher-order curvature invariants are coupled to scalar (dilaton/modulus) fields. One might try to fix these fields by invoking some non-perturbative mechanism. In that case, the GB term does not contribute to the four-dimensional equations of motion. However, the higher-order curvature terms contribute in this case. Their presence crucially modifies the fate of a phantom dark energy universe. Since it is difficult to realize any scenario with a fixed dilaton or modulus field, the analysis involving dynamically evolving fields becomes very important. A number of papers [11–24] are devoted to the possibility of having a dark energy with a GB term and a dilaton/modulus field with a non-trivial potential. A steep exponential potential exhibits a scaling solution that mimics the background (matter/radiation); the nucleosynthesis constraint is satisfied provided the slope of the potential is large [25]. The scaling solution describes a decelerating universe. It is surprising that the GB term can cause a transition from the matter dominated era to a dark energy universe and it can also lead to transient phantom energy, provided the slope of the exponential potential and the dilaton coupling to the GB invariant are chosen properly [25], or a more complicated choice of scalar potentials is done [26]. In such a scenario with the exponential potential, it is quite difficult to satisfy the nucleosynthesis constraint and, secondly, the coupling also becomes very large. Since the introduction of the dilaton potential needs assumptions about some non-perturbative mechanisms and the massless dilaton naturally arises in the string loop expansion of the low energy effective theory, it is important to explore the possibility of a dark energy solution with a massless dilaton. To this effect, the second order curvature correction was considered in [12]. This next-to-leading correction contains a higher-order Euler density that identically vanishes for space-time dimensions less than six; the other remaining term is a curvature invariant of order three. The model can lead to a stable dark energy solution. It is interesting to note that the third-order correc-

tion in α' crucially depends on the type of string theory model. In this paper we incorporate string loop corrections up to order three in α' to the Einstein–Hilbert action with a massless dilaton. We investigate the cosmological dynamics of the model and explore whether a particular string type is actually sensitive to the existence of dark energy. We also outline a general prescription of the construction of the de Sitter solution in presence of higher-order curvature invariants coupled to the dilaton field.

The paper is organized as follows. In Sect. 2, we set up the general evolution equations from a string effective Lagrangian that incorporates curvature corrections, up to order four in R, coupled to a dynamically evolving massless dilaton. In Sect. 3, we explore the viability of a dark energy solution for models based upon type II, heterotic and bosonic strings in the framework of a perturbative string theoretic set up. Section 4 is devoted to the stability analysis of the de Sitter solution in the case of a bosonic string model. In Sect. 5, we present a reconstruction program for a general action with higher-order curvature invariants coupled to dilaton functions. Section 6 outlines the construction of the de Sitter solution in the general case. Section 7 presents our conclusions and an outlook.

2 Evolution equations

The process of compactification of the string theory from higher to four dimensions introduces scalar (moduli/dilaton) fields that are coupled to curvature invariants. For simplicity, we shall neglect the moduli fields associated with the radii of the internal space. In what follows, we consider the low-energy effective string theory action [10, 13]

$$\mathcal{S} = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{R}{2} + \mathcal{L}_\phi + \mathcal{L}_c + \ldots \right], \qquad (1)$$

where ϕ denotes the dilaton field, which is related to the string coupling, R is the scalar curvature, $\mathcal{L}_\phi$ denotes the scalar field Lagrangian, and $\mathcal{L}_c$ encodes the string curvature correction term to the Einstein–Hilbert: action [10]

$$\mathcal{L}_\phi = -\partial_\mu \phi \partial^\mu \phi - V(\phi), \qquad (2)$$

$$\mathcal{L}_c = c_1 \alpha' e^{2\frac{\phi}{\phi_0}} \mathcal{L}_c^{(1)} + c_2 \alpha'^2 e^{4\frac{\phi}{\phi_0}} \mathcal{L}_c^{(2)} + c_3 \alpha'^3 e^{6\frac{\phi}{\phi_0}} \mathcal{L}_c^{(3)}, \quad (3)$$

where α' is the string expansion parameter, $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$, and $\mathcal{L}_c^{(3)}$ describe the leading order (GB term), the second-order and third-order curvature corrections, respectively. The terms $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$ and $\mathcal{L}_c^{(3)}$ in the Lagrangian have the following form:

$$\mathcal{L}_c^{(1)} = \Omega_2, \qquad (4)$$

$$\mathcal{L}_c^{(2)} = 2\Omega_3 + R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\lambda\rho} R^{\lambda\rho}_{\mu\nu}, \qquad (5)$$

$$\mathcal{L}_c^{(3)} = \mathcal{L}_{31} - \delta_H \mathcal{L}_{32} - \frac{\delta_B}{2} \mathcal{L}_{33}. \qquad (6)$$

Here $\delta_{\rm B}, \delta_{\rm H} = 0, 1$ and

$$\Omega_2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,, \tag{7}$$

$$\Omega_3 \propto \epsilon^{\mu\nu\rho\sigma\tau\eta}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\eta'}R_{\mu\nu}^{\mu'\nu'}R_{\rho\sigma}^{\rho'\sigma'}R_{\tau\eta}^{\tau'\eta'}\,, \tag{8}$$

$$\mathcal{L}_{31} = \zeta(3)R_{\mu\nu\rho\sigma}R^{\alpha\nu\rho\beta}\big(R_{\delta\beta}^{\mu\gamma}R_{\alpha\gamma}^{\delta\sigma} - 2R_{\delta\alpha}^{\mu\gamma}R_{\beta\gamma}^{\delta\sigma}\big)\,, \tag{9}$$

$$\begin{aligned}
\mathcal{L}_{32} &= \frac{1}{8}\big(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\big)^2 + \frac{1}{4}R_{\mu\nu}^{\gamma\delta}R_{\gamma\delta}^{\rho\sigma}R_{\rho\sigma}^{\alpha\beta}R_{\alpha\beta}^{\mu\nu}\\
&\quad - \frac{1}{2}R_{\mu\nu}^{\alpha\beta}R_{\alpha\beta}^{\rho\sigma}R_{\sigma\gamma\delta}^{\mu}R_{\rho}^{\nu\gamma\delta} - R_{\mu\nu}^{\alpha\beta}R_{\alpha\beta}^{\rho\nu}R_{\rho\sigma}^{\gamma\delta}R_{\gamma\delta}^{\mu\sigma}\,,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathcal{L}_{33} &= \big(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\big)^2 - 10R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\sigma}R_{\sigma\gamma\delta\rho}R^{\beta\gamma\delta\rho}\\
&\quad - R_{\mu\nu\alpha\beta}R_{\sigma}^{\mu\nu\rho}R^{\beta\sigma\gamma\delta}R_{\delta\gamma\rho}^{\alpha}\,.
\end{aligned} \tag{11}$$

The correction terms are different depending on the type of string theory; the dependence is encoded in the curvature invariants and in the coefficients (c_1, c_2, c_3) and $\delta_{\rm H}, \delta_{\rm B}$.

- For the type II superstring theory: $(c_1, c_2, c_3) = (0, 0, 1/8)$ and $\delta_{\rm H} = \delta_{\rm B} = 0$.
- For the heterotic superstring theory: $(c_1, c_2, c_3) = (1/8, 0, 1/8)$ and $\delta_{\rm H} = 1, \delta_{\rm B} = 0$.
- For the bosonic superstring theory: $(c_1, c_2, c_3) = (1/4, 1/48, 1/8)$ and $\delta_{\rm H} = 0, \delta_{\rm B} = 1$.

The higher-order curvature corrections look complicated, in general. However, the analysis become tractable in the case of a Friedmann–Robertson–Walker universe. In what follows we will specialize to the case of the FRW metric with a lapse function $N(t)$, namely,

$$ds^2 = -N(t)^2\,dt^2 + a(t)^2\sum_{i=1}^{3}(dx^i)^2\,. \tag{12}$$

In the FRW background, the leading and next-to-leading corrections simplify; they depend upon the lapse function and its time derivative $\dot{N}$. Since only terms linear in $\dot{N}$ contribute to the evolution equations, we shall omit the higher powers of the time derivative of the lapse function. We then have the following expressions for the curvature invariants:

$$\mathcal{L}_{\rm c}^{(1)} = \frac{24}{N^4}H^2 I - \frac{24\dot{N}}{N^6}H^3\,, \tag{13}$$

$$\mathcal{L}_{\rm c}^{(2)} = \frac{24}{N^6}(H^6 + I^3) - \frac{72\dot{N}}{N^7}HI^2\,, \tag{14}$$

and

$$\begin{aligned}
\mathcal{L}_{31} &= -\frac{6\zeta(3)}{N^8}(3H^8 + 4H^4I^2 + 4H^2I^3 + I^4)\\
&\quad + \frac{6\zeta(3)\dot{N}}{N^9}(8H^5I + 12H^3I^2 + 4HI^3)\,,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\mathcal{L}_{32} &= -\frac{6}{N^8}(5H^8 + 2H^4I^2 + 5I^4)\\
&\quad + \frac{6\dot{N}}{N^9}(4H^5I + 20HI^3)\,,
\end{aligned} \tag{16}$$

$$\begin{aligned}
\mathcal{L}_{33} &= -\frac{6}{N^8}(60H^8 + 32H^4I^2 + 60I^4)\\
&\quad + \frac{6\dot{N}}{N^9}(64H^5I + 240HI^3)\,.
\end{aligned} \tag{17}$$

Here $I = \dot{H} + H^2$.

Varying the action (1) with respect to N and $\dot{N}$, we find the modified Friedmann equation:

$$3H^2 = \rho_{\rm c} + \rho_\phi\,, \tag{18}$$

where

$$\begin{aligned}
\rho_{\rm c} &= \sum_{m=1}^{3}\bigg\{\dot{\xi}_m(\phi)\bigg(\frac{\partial\mathcal{L}_{\rm c}^{(m)}}{\partial\dot{N}}\bigg) + \xi_m(\phi)\bigg(\frac{d}{dt}\bigg(\frac{\partial\mathcal{L}_{\rm c}^{(m)}}{\partial\dot{N}}\bigg)\\
&\quad + 3H\frac{\partial\mathcal{L}_{\rm c}^{(m)}}{\partial\dot{N}} - \frac{\partial\mathcal{L}_{\rm c}^{(m)}}{\partial N} - \mathcal{L}_{\rm c}^{(m)}\bigg)\bigg\}\bigg|_{N=1}\,,
\end{aligned} \tag{19}$$

and where $\xi_1(\phi)$, $\xi_2(\phi)$ and $\xi_3(\phi)$ come from the first-, second- and third-order correction terms, respectively, and can be written as

$$\xi_m(\phi) = \alpha'^m e^{2m\frac{\phi}{\phi_0}}\,, \quad m = 1, 2, 3\,. \tag{20}$$

Let us consider the scalar field equation of motion derived by varying the action (1) keeping in mind the perturbative string theoretic description $(V(\phi) = 0)$

$$\ddot{\phi} + 3H\dot{\phi} - \xi_1'\mathcal{L}_{\rm c}^{(1)} - \xi_2'\mathcal{L}_{\rm c}^{(2)} - \xi_3'\mathcal{L}_{\rm c}^{(3)} = 0\,. \tag{21}$$

The evolution equations look complicated, in general (see (A.1) and (A.2) in the appendix), and the analysis of the cosmological dynamics seems to be a difficult task. We, therefore, take a different route in the search of a dark energy solution. We shall consider the following simple solution [11] and examine its viability in the present case:

$$H = \frac{h_0}{t}\,, \quad \phi = \phi_0\ln\frac{t}{t_1}\,,$$

for $h_0 > 0$, and

$$H = -\frac{h_0}{t_s - t}\,, \quad \phi = \phi_0\ln\frac{t_s - t}{t_1}\,, \tag{22}$$

when $h_0 < 0$ with t_1 as an undetermined constant. This solution leads to a constant equation of state

$$w_{\rm eff} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \frac{2}{3h_0}\,, \tag{23}$$

which corresponds to dark energy (respectively phantom dark energy) for $h_0 > 0$ (respectively $h_0 < 0$), de Sitter solution is obtained for $h_0 \to \infty$.

We will next analyze in detail whether the evolution equations, (A.1) and (A.2), exhibit the given solution (22) for realistic values of the constants h_0, t_1, and ϕ_0. By substituting (22) into (A.1) and (A.2), we obtain the algebraic equations

$$-\phi_0^2 + 3h_0\phi_0^2 + f_1(h_0)X + f_2(h_0)X^2 + f_3(h_0)X^3 = 0 \tag{24}$$

and

$$\frac{\phi_0^2}{2} - 3h_0^2 + f_4(h_0)X + f_5(h_0)X^2 + f_6(h_0)X^3 = 0\,, \tag{25}$$

where $X \equiv \alpha'/t_1^2$ and f'^s are given by the following algebraic expressions:

$$f_1(h_0) = \delta_{\mathrm{HB}}\left(12h_0^3 - 12h_0^4\right),$$

$$f_2(h_0) = \delta_{\mathrm{B}}\left(2h_0^3 - 6h_0^4 + 6h_0^5 - 4h_0^6\right),$$

$$f_3(h_0) = \zeta(3)\left(\frac{9}{2}h_0^4 - 36h_0^5 + 99h_0^6 - 108h_0^7 + 54h_0^8\right)$$
$$+ \delta_{\mathrm{H}}\left(-\frac{45}{2}h_0^4 + 90h_0^5 - 144h_0^6 + 108h_0^7 - 54h_0^8\right)$$
$$+ \delta_{\mathrm{B}}\left(-135h_0^4 + 540h_0^5 - 882h_0^6 + 684h_0^7 - 342h_0^8\right),$$

$$f_4(h_0) = \delta_{\mathrm{HB}}\left(-12h_0^3\right),$$

$$f_5(h_0) = \delta_{\mathrm{B}}\left(-h_0^3 + 3h_0^5 + \frac{1}{2}h_0^6\right),$$

$$f_6(h_0) = \zeta(3)\left(-\frac{9}{4}h_0^4 + 15h_0^5 - \frac{57}{2}h_0^6 + 9h_0^7 - 9h_0^8\right)$$
$$+ \delta_{\mathrm{H}}\left(\frac{45}{4}h_0^4 - 15h_0^5 - 15h_0^6 + 36h_0^7 + 9h_0^8\right)$$
$$+ \delta_{\mathrm{B}}\left(\frac{135}{2}h_0^4 - 90h_0^5 - 75h_0^6 + 198h_0^7 + 57h_0^8\right),$$

(26)

where $\delta_{\mathrm{HB}} = 0, 1/2, 1$ for type II, heterotic and bosonic string, respectively.

In what follows we would like to analyze the validity of (24) and (25) for realistic values of the constants ϕ_0 and t_1, corresponding to specific values of h_0 relevant to dark energy observations.

3 Dark energy solution

We shall first examine the existence of dark energy solutions (22) in general and then will specialize to particular types of string models. We will be interested in finding out whether dark energy can distinguish amongst the string types. The case of the bosonic string will be of special interest.

3.1 The general case

We can combine (24) and (25) into a single cubic equation as

$$A_3(h_0)X^3 + A_2(h_0)X^2 + A_1(h_0)X - 6h_0^2(1 - 3h_0) = 0,$$

(27)

where the coefficients of X are given by

$$A_m(h_0) = f_m(h_0) + 2(1 - 3h_0)f_{3+m}(h_0), \quad m = 1, 2, 3.$$

(28)

In the case of $m \leq 3$, we always have the analytic formulae for the roots, which will be useful for the interpretation of the relation and the contribution from each of the correction terms to the solution.

The positivity of X, the real root of the cubic equation will impose a restriction on the possible values of h_0. The real solution for (27) can be obtained from the cubic root formula:

$$X(h_0) = s_1(h_0) + s_2(h_0) - \frac{1}{3}\frac{A_2(h_0)}{A_3(h_0)},$$

(29)

where

$$s_1(h_0) = [r + (q^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}},$$

$$s_2(h_0) = [r - (q^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}},$$

$$r(h_0) = \frac{3h_0^2(1 - 3h_0)}{A_3(h_0)} + \frac{1}{6}\frac{A_1(h_0)A_2(h_0)}{A_3(h_0)^2} - \frac{1}{27}\left(\frac{A_2(h_0)}{A_3(h_0)}\right)^3,$$

$$q(h_0) = \frac{1}{3}\frac{A_1(h_0)}{A_3(h_0)} - \frac{1}{9}\left(\frac{A_2(h_0)}{A_3(h_0)}\right)^2.$$

(30)

3.1.1 $X(h_0) \equiv \alpha'/t_1^2 > 0$

The cubic equation has one real root X provided $q^3 + r^2 > 0$. We have checked numerically that the relation $q^3 + r^2 > 0$ is true for all h_0 in the region $0.8 < |h_0|$, for all three string types. Equation (29) should then be used in order to find the range of h_0 such that $X(h_0) > 0$ and the corresponding equation of state parameter w_{eff} be confronted with the observations.

3.1.2 $\Omega_{\mathrm{c}} < 1$

Another important constraint on h_0 is dictated by the fact that $\phi_0^2 < 1$, as the dilaton is a real scalar function. The cubic equation (27) does not involve ϕ_0; it enters into the Friedmann equation through ρ_{c}, which encodes all higher-order curvature corrections. Using the Friedmann equation (18) and (22), we find

$$3h_0^2 = \rho_\phi + \rho_{\mathrm{c}},$$

$$1 = \frac{\rho_\phi}{3h_0^2} + \frac{\rho_{\mathrm{c}}}{3h_0^2} \equiv \frac{\phi_0^2}{6h_0^2} + \frac{\rho_{\mathrm{c}}}{3h_0^2},$$

$$1 = \Omega_\phi + \Omega_{\mathrm{c}},$$

(31)

where Ω_{c} is the dimensionless density parameter contributed by the correction terms. The constraint $\Omega_{\mathrm{c}} < 1$ is equivalent to $\phi_0^2 > 0$, otherwise the dilaton would turn complex and this would put a bound on the possible range of h_0 which should be combined with the constraint dictated by the positivity of real root of the cubic equation (27). The range of h_0 compatible with the two constraints should then be confronted with the observation on the equation of state for the dark energy. The recent analysis of the three-year WMAP data combined with the supernova legacy survey (SNLS) constrains the dark energy equation of state parameter w_{DE}. At 68% confidence level, the best fit value w_{DE} is given by $w_{\mathrm{DE}} = -1.06^{+0.07}_{-0.08}$. If the flat prior is imposed, the parameter is constrained by $w_{\mathrm{DE}} = -0.97^{+0.07}_{-0.09}$, which translates into a bound on h_0, as $h_0 \leq -11.11$ and $h_0 \geq 6.67$.

We next turn to the individual string models to find out their viability as dark energies, in view of the aforesaid constraints.

3.2 Type II string

The case of a type II string is simplest to investigate. In this case, we have $A_1(h_0) = A_2(h_0) = 0$ and (27) reduces to

$$A_3(h_0)X(h_0)^3 - 6h_0^2(1-3h_0) = 0 \,, \tag{32}$$

which has the following solution:

$$X(h_0) = (2r(h_0))^{\frac{1}{3}} = \left(\frac{6h_0^2(1-3h_0)}{A_3(h_0)} \right)^{\frac{1}{3}} \,. \tag{33}$$

In this case the expression of A_3 simplifies to

$$A_3(h_0) = f_3(h_0) + 2(1-3h_0)f_6(h_0) \,, \tag{34}$$

with

$$f_3(h_0) = \zeta(3)\left(\frac{9}{2}h_0^4 - 36h_0^5 + 99h_0^6 - 108h_0^7 + 54h_0^8 \right) \,,$$
$$f_6(h_0) = \zeta(3)\left(-\frac{9}{4}h_0^4 + 15h_0^5 - \frac{57}{2}h_0^6 + 9h_0^7 - 9h_0^8 \right) \,. \tag{35}$$

Let us first implement the condition $X(h_0) = \alpha'/t_1^2 > 0$. The sign of A_3 is important for constraining h_0 using the positivity of X. From (35), we obtain

$$A_3(h_0) = \zeta(3)\left(\frac{15}{2}h_0^5 - 48h^6 + 81h^7 - 18h_0^8 + 54h_0^9 \right) \,.$$

We have plotted A_3 in Fig. 1. The plot shows that $A_3(h_0) > 0$ for $h_0 > 0$ and $A_3(h_0) < 0$ when $h_0 < 0$. Using (33), we get the possible region that gives $X(h_0) > 0$ as $0 < h_0 < 1/3$ which always yields $w_{\text{eff}} > 1$. Thus, no viable solution exists in this case. Therefore, up to 4th order corrections in R, the type II superstring model is clearly ruled out as dark energy (for the $\Omega_c < 1$ case, see Fig. 2).
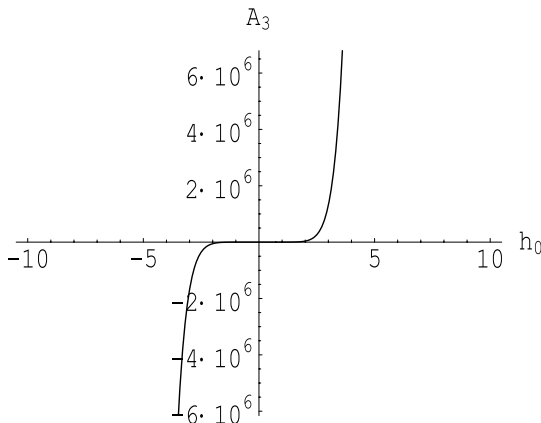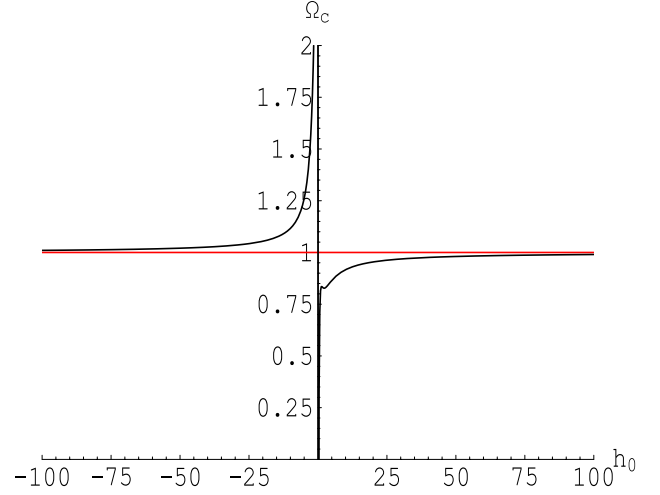
Fig. 2. A plot of Ω_c against h_0 for the type II string shows that for $h_0 > 0$, the density parameter $\Omega_c < 1$

3.3 Heterotic string

In this case $A_2(h_0) = 0$, and $A_1(h_0)$ and $A_3(h_0)$ are given by

$$A_3(h_0) = f_3(h_0) + 2(1-3h_0)f_6(h_0) \,,$$
$$A_1(h_0) = f_1(h_0) + 2(1-3h_0)f_4(h_0) \,,$$

where

$$A_3(h_0) = -\frac{15}{2}h_0^5 - 84h_0^6 + 270h_0^7 - 252h_0^8 - 54h_0^9$$
$$+ \zeta(3)\left(\frac{15}{2}h_0^5 - 48h_0^6 + 81h_0^7 - 18h_0^8 + 54h_0^9 \right)$$
$$\approx 1.515h_0^5 - 141.699h_0^6 + 367.367h_0^7$$
$$- 273.637h_0^8 + 10.911h_0^9 \,. \tag{36}$$

In this case, we check the consistency of $X(h_0)$ and Ω_c numerically as (33) does not seem to hold in this case. How-

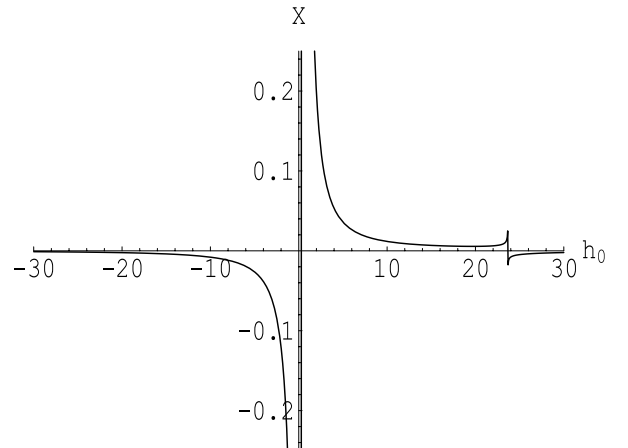**Fig. 1.** A plot of A_3 against h_0 is shown for type II string model

Fig. 3. A plot of X against h_0 shows that $0 < h_0 < 23.68$ is the viable range of h_0 for the heterotic string
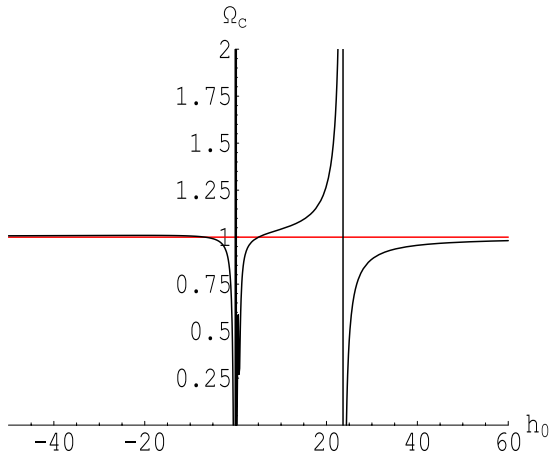
Fig. 4. A plot of Ω_c versus h_0 for the heterotic string model. The region given by $\Omega < 1$ shows the allowed range of h_0, which corresponds to $0.8 < h_0 \leq 5.04$ and $h_0 > 23.68$
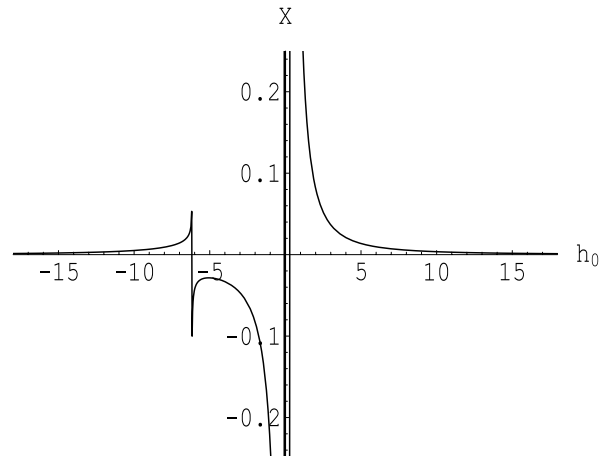

Fig. 5. The plot of X versus h_0 for the bosonic string shows that $X(h_0) > 0$ for $h_0 < -6.189$ or $h_0 > 0$

ever, as we remark below, A_3 might still be used as a yard stick for the consistency check.

We plot $X(h_0)$ in Fig. 3, which shows that $X > 0$ provided $0 < h_0 < 23.68$. This constraint should be combined with $\Omega_c < 1$. The plot of Ω_c in Fig. 4 tells us that either $0 < h_0 < 5.04$ or $h_0 > 23.68$. Thus, the allowed range for the parameter h_0 is $0 < h_0 < 5.04$, which corresponds to $w_{\text{eff}} \geq -0.868$. Such a value of the equation of state parameter is ruled out by recent $WMAP_3$ and SNLS survey data. However, the combined data $(\text{CMB} + \text{LSS} + \text{SNLS})$ force the equation of state to vary as $-1.001 < w_{\text{DE}} < -0.875$. This results shows that the heterotic string model is marginally compatible with dark energy observations.

3.4 Bosonic string

We now turn to the bosonic string, for which $A_m \neq 0$ for $m = 1, 2, 3$. All f^s contribute to A_m in this case. We quote the expression for A_3:

$$
\begin{aligned}
A_3(h_0) = & -45h_0^5 - 492h_0^6 + 1530h_0^7 - 1416h_0^8 - 342h_0^9 \\
& + \zeta(3)\left(\frac{15}{2}h_0^5 - 48h_0^6 + 81h_0^7 - 18h_0^8 + 54h_0^9\right) \\
\approx & -35.985h_0^5 - 549.699h_0^6 + 1627.370h_0^7 \\
& - 1437.640h_0^8 - 277.089h_0^9.
\end{aligned}
\tag{37}
$$

In order to the check if $X(h_0) > 0$ and $\Omega_c < 1$, we display our numerical results in Figs. 5 and 6, which show that

a. $X(h_0) > 0$ for $h_0 < -6.189$ or $h_0 > 0$;
b. $\Omega_c < 1$ for $h_0 > 0.5$.

Note that $(r^2 + q^3) > 0$, the condition for the existence of one real root of (27), constrains h_0 to be $h_0 > 0.8$. We therefore conclude that the allowed range for h_0 is given by $h_0 > 0.8$ and this corresponds to $-1 < w_{\text{eff}} < -0.17$. The requirement for the dilaton to be real clearly excludes the possibility of phantom energy. It is really interesting that the bosonic string responds positively to the requirement of dark energy.
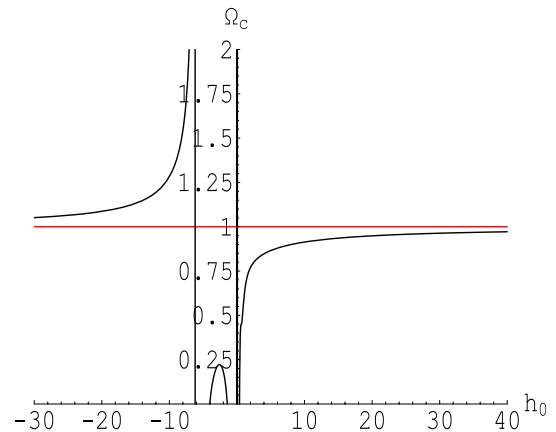

Fig. 6. Ω_c is plotted here against h_0 for the case of the bosonic string

Before moving to the next section, a remark about A_3 is in order. In the case of a type II superstring, A_1 and A_2 vanish identically, leading to $s_2 = 0$. The sign of A_3 then becomes important for the consistency check on $X(h_0)$. In the cases of heterotic and bosonic strings this is no longer true. In these cases we have directly checked the positivity of $X(h_0)$ using numerical treatments. Interestingly enough, we have found numerically that, for a generic range of the parameter h_0, it turns out that $s_1 \gg s_2$ telling us that (33) still holds approximately for numerical values of h_0 that are of interest to us. We then could analyze the bosonic and heterotic models by checking the sign of A_3 as we did for the case of the type II string. In fact, our numerical check shows that we reproduce the exact numerical results presented here to a good accuracy.

4 The de Sitter solution and its stability

It is only in the bosonic case that we have the desired solution (normal dark energy). The general expressions for

the Friedmann equation and the scalar field equation of motion, given in the appendix, are difficult to analyze in general. However, in the de Sitter case the equations get simplified, and in what follows we will analyze the stability of this solution.

Following Nojiri et al. [11] we define two new variables:

$$X = \frac{\dot{\phi}}{H}, \quad Y = H^2 \alpha' e^{2\phi/\phi_0}. \tag{38}$$

With these new variables the Friedmann equation and the equation of motion for the scalar field (see appendix) can be written as

$$\frac{dX}{dN} = -3X + \frac{12}{\phi_0}Y + \frac{4}{\phi_0}Y^2 + \frac{342}{\phi_0}Y^3 - \frac{54\zeta(3)}{\phi_0}Y^3, \tag{39}$$

$$\frac{dY}{dN} = -\frac{1}{2} + \frac{1}{12}(X^2 + Y^2) + \frac{19}{2}Y^3 - \frac{1}{\phi_0}XY^2$$
$$- \frac{144}{\phi_0}XY^3 - 3\zeta(3)Y^3\left(\frac{1}{2} - \frac{6X}{\phi_0}\right). \tag{40}$$

These expressions are much simpler, since H is a constant and its derivatives vanish identically. For $\phi_0 = -0.01$ the fixed points (X_c, Y_c) are

$X_c \rightarrow$	$Y_c \rightarrow$
-25.1415	0.0574
-2.4860	0.0062
2.4860	-0.0062
29.4780	-0.0680
$-0.0005 \pm 0.0060i$	$-0.0072 \pm 0.2080i$

The first critical point in the table above is relevant for us since the second turns out to be unstable (see Fig. 7). The third and the fourth point give a negative value to

the string expansion parameter and are, therefore, not relevant. We next examine the stability of the solution around the critical point $(X_c, Y_c) = (-25.1415, 0.0574)$. The perturbation matrix $\mathcal{M}$ has the following form:

$$\mathcal{M} = \begin{pmatrix} -3 \\ \frac{X_c}{6} - \frac{Y_c^2}{\phi_0}(1 + [114 - 18\zeta(3)]Y_c) \end{pmatrix}$$

$$\frac{1}{\phi_0}\left(12 + 8Y_c + [1026 - 162\zeta(3)]Y_c^2\right)$$
$$\frac{Y_c}{6} + \frac{Y_c^2}{2}[57 - 9\zeta(3)] - \frac{2X_cY_c}{\phi_0}(1 + [171 - 27\zeta(3)]Y_c) \end{pmatrix}.$$

The eigenvalues of the stability matrix $\mathcal{M}$ are: -2161.75 and -1.7903. Therefore, the critical point is a stable node. In general fixed points exist for the range $|\phi_0| \in (0, 0.05882)$.

5 Reconstruction of the universe expansion history: beyond the low-energy string effective action

In this section we study a more general gravitational action, in which the coefficients of the curvature corrections depend on the dilaton. In addition, a scalar potential is added. We then consider the reconstruction of such general modified gravity from the universe expansion history, following the technique developed in [27]. Let us keep $\xi_m(\phi)$ to be general functions of the scalar field ϕ. For simplicity, however, we neglect $\mathcal{L}_c^{(3)}$. Then, the action has the following form:

$$\mathcal{S} = \int d^4x \sqrt{-g}$$
$$\times \left[\frac{R}{2} - \partial_\mu\phi\partial^\mu\phi - V(\phi) + \xi_1(\phi)\mathcal{L}_c^{(1)} + \xi_2(\phi)\mathcal{L}_c^{(2)}\right]. \tag{41}$$
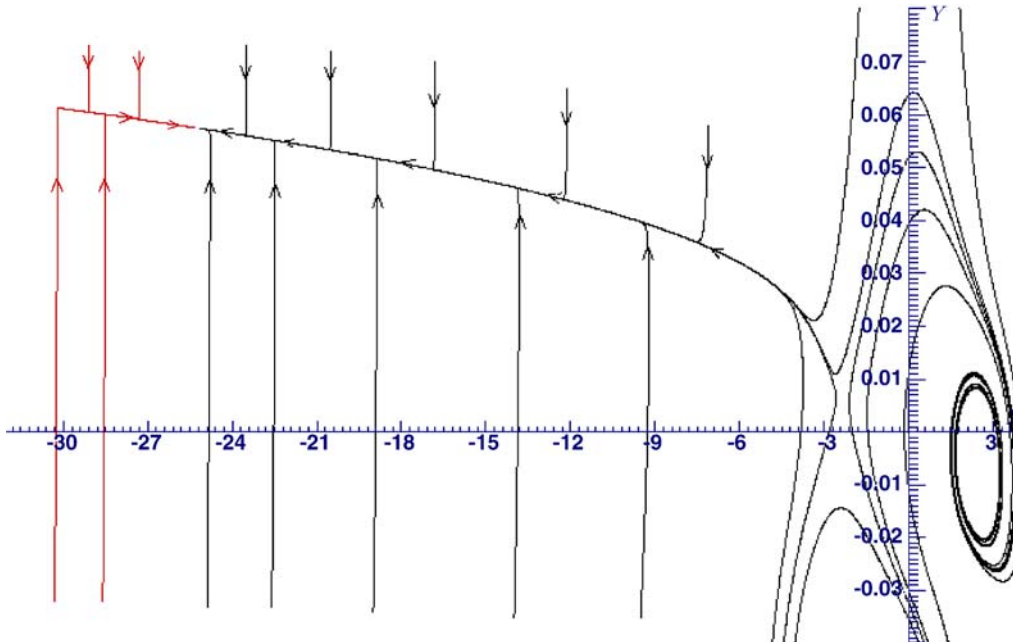

Fig. 7. The phase portrait of cosmological evolution described by (1) in the case of the bosonic string restricted to the de Sitter case for $\phi_0 = -0.01$. Trajectories starting anywhere in the phase space converge at the stable node $(-25.1415, 0.0574)$

Even if one includes $\mathcal{L}_c^{(3)}$, we can reconstruct the action, although the final expression becomes rather complicated. Neglecting $\mathcal{L}_c^{(3)}$, the explicit form of the FRW equation and the scalar field equation follows:

$$
\begin{aligned}
0 = &-3H^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) - 24\dot{\xi}_1(\phi)H^3 \\
&- 72\dot{\xi}_2(\phi)H(H^2 + \dot{H})^2 \\
&+ 24\xi_2(\phi)\{-6H(H^2+\dot{H})\ddot{H} \\
&+ 2(H^2+\dot{H})^3 - 18H^2(H^2+\dot{H})^2 + 5H^6\}, \quad (42)
\end{aligned}
$$

$$
\begin{aligned}
0 = &\ddot{\phi} + 3H\dot{\phi} + V'(\phi) - 24\xi_1'(\phi)(H^2+\dot{H}) \\
&+ 24\xi_2'(\phi)(H^6 + (H^2+\dot{H})^3). \quad (43)
\end{aligned}
$$

By combining (42) and (43), we obtain

$$
\xi_1(\phi(t)) = \int dt \frac{a(t)W(t)}{H^2}, \quad (44)
$$

$$
\begin{aligned}
V(\phi(t)) = &3H^2 - \frac{1}{2}\dot{\phi}^2 + 24Ha(t)W(t) \\
&+ 72\dot{\xi}_2(\phi(t))H(H^2+\dot{H}) \\
&+ 24\xi_2(\phi(t))\left\{6(H^2+\dot{H})H\ddot{H} - 2(H^2+\dot{H})^3 \right. \\
&\left. + 18(H^2+\dot{H})^2H^2 - 5H^6\right\}, \quad (45)
\end{aligned}
$$

$$
\begin{aligned}
W(t) = &\int \frac{dt}{a(t)}\left[-\frac{\dot{H}}{4} - \frac{\dot{\phi}^2}{8}\right. \\
&+ \dot{\xi}_2(\phi(t))\left\{-13H^5 - 45H^3\dot{H} - 3H\dot{H}^2\right. \\
&\left. + \frac{\dot{H}^3}{H} - 12(H^2+\dot{H})\ddot{H}\right\} \\
&+ 6\xi_2(\phi)\{-11H^4\dot{H} - 20H^2\dot{H}^2 - 4\dot{H}^3 \\
&\left. + (-5H^3 - 7H\dot{H} - \ddot{H})\ddot{H} - (H^2+\dot{H})\dddot{H}\}\right]. \\
&\quad (46)
\end{aligned}
$$

Hence, if we consider the theory where $\xi_1(\phi)$ and $V(\phi)$ are expressed in terms of two functions, $g(t)$ and $f(\phi)$, and an arbitrary $\xi_2(\phi)$ as follows (compare with a reconstruction in the less complicated case when only the first order correction is present [12, 26, 28])

$$
\xi_1(\phi) = \int dt \frac{e^{g(f(\phi))}U(\phi)}{g'(f(\phi))^2}, \quad (47)
$$

$$
\begin{aligned}
V(\phi) = &3g'(f(\phi))^2 - \frac{1}{2f'(\phi)^2} + 24g'(f(\phi))e^{g(f(\phi))}U(\phi) \\
&+ \frac{72\xi_2(\phi)g'(f(\phi))}{f'(\phi)}(g'(f(\phi))^2 + g''(f(\phi))) \\
&+ 24\xi_2(\phi) \\
&\times \{6(g'(f(\phi))^2 + g''(f(\phi)))g'(f(\phi))g'''(f(\phi)) \\
&- 2(g'(f(\phi))^2 + g''(f(\phi)))^3 \\
&+ 18(g'(f(\phi))^2 + g''(f(\phi)))^2 g'(f(\phi))^2 \\
&- 5g'(f(\phi))^6\}, \quad (48)
\end{aligned}
$$

$$
\begin{aligned}
U(\phi) = &\int \frac{f'(\phi)\,d\phi}{e^{g(f(\phi))}}\left[-\frac{g'(f(\phi))}{4} - \frac{1}{8f'(\phi)^2}\right. \\
&+ \frac{\xi_2'(\phi)}{f'(\phi)}\left\{-13g'(f(\phi))^5 - 45g'(f(\phi))^3 g''(f(\phi))\right. \\
&- 3g'(f(\phi))(g''(f(\phi)))^2 + \frac{g''(f(\phi))^3}{g'(f(\phi))} \\
&\left. - 12(g'(f(\phi))^2 + g''(f(\phi)))g'''(f(\phi))\right\} \\
&+ 6\xi_2(\phi)\left\{-11g'(f(\phi))^4 g''(f(\phi))\right. \\
&- 20g'(f(\phi))^2 g''(f(\phi))^2 - 4g''(f(\phi))^3 \\
&+ (-5g'(f(\phi))^3 - 7g'(f(\phi))g''(f(\phi)) \\
&- g'''(f(\phi)))g'''(f(\phi)) \\
&\left.\left. - (g'(f(\phi))^2 + g''(f(\phi)))g''''(f(\phi))\right\}\right], \quad (49)
\end{aligned}
$$

then it is not hard to check that a solution is given by

$$
H = g(t), \quad \phi = f^{-1}(t). \quad (50)
$$

Here $f^{-1}(t)$ is the inverse function of $f(\phi)$.

An example of this situation is the following:

$$
g(t) = H_0 t + H_1 \ln\left(\frac{t}{t_0}\right), \quad (51)
$$

and after properly defining $f(\phi)$, we obtain

$$
H(t) = H_0 + \frac{H_1}{t}. \quad (52)
$$

When t is small, H (52) behaves as that in a universe with a perfect fluid, with $w_{\text{eff}} = -1 + 2/3H_1$, and when t is large, H behaves as in the de Sitter space, where H is a constant. Then, if we choose $H_1 = 2/3$, we find that before the acceleration epoch, the universe behaves as a matter dominated one with $w_{\text{eff}} = 0$. After that, it enters the acceleration phase.

Another example is

$$
g(t) = \widetilde{H}_0 \ln\frac{t}{t_0} - \widetilde{H}_1 \ln\left(\frac{t_0 - t}{t_0}\right), \quad (53)
$$

which gives

$$
H(t) = \frac{\widetilde{H}_0}{t} + \frac{\widetilde{H}_1}{t_0 - t}. \quad (54)
$$

Here $\widetilde{H}_0$, $\widetilde{H}_1$, and t_0 are positive constants. When t is small, H in (54) behaves in a way corresponding to the perfect fluid case, with $w_{\text{eff}} = -1 + 2/3\widetilde{H}_0$. Then, if we choose $\widetilde{H}_0 = 2/3$, the matter dominated universe occurs. On the other hand, when $t \sim t_0$ is large, H behaves as in the phantom universe with $w_{\text{eff}} = -1 - 2/3\widetilde{H}_1 < -1$ and a big rip singularity at $t = t_0$ will appear. The three-year WMAP data are analyzed in [29], which shows that the combined analysis of WMAP with the SNLS constrains the dark energy equation of state w_{DE}, pushing it clearly towards the cosmological constant value. The marginalized best fit values of the equation of state parameter at 68%

confidence level are given by $-1.14 \leq w_{\mathrm{DE}} \leq -0.93$, which corresponds to $\widetilde{H}_1 > 10.7$ as $\widetilde{H}_1$ is positive. In the case when one takes as a prior that the universe is flat, the combined data gives $-1.06 \leq w_{\mathrm{DE}} \leq -0.90$, which corresponds to $\widetilde{H}_1 > 25.0$. Therefore, the possibility that $w_{\mathrm{DE}} < -1$ has not been excluded.

Finally, an additional example is the ΛCDM-type cosmology:

$$g(t) = \frac{2}{3(1+w)} \ln \left[\alpha \sinh \left(\frac{3(1+w)}{2l} (t - t_0) \right) \right],$$
$$\alpha^2 \equiv \frac{1}{3} l^2 \rho_0 a_0^{-3(1+w)} . \tag{55}$$

Here l is the length scale given by the cosmological constant $l \sim (10^{-33}\,\mathrm{eV})^{-1}$ and t_0 is a constant. The time-development of the universe given by $g(t)$ (55) can be realized in the usual Einstein gravity with a cosmological constant Λ and cold dark matter (CDM), which could be regarded as dust. The corresponding scalar potentials can indeed be written explicitly, but they are quite complicated functions.

6 Construction of the de Sitter solution for a general effective action

Let us study the possibility of realizing de Sitter space from the scalar field equation and the Friedmann equation. The coefficients c_1, c_2, c_3, and also δ_{H}, δ_{B} in the action (3)–(6) depend on what kind of string theory – that is, bosonic string, type II superstring theory, or heterotic string theory – we are considering. Furthermore, these coefficients could depend on the details of the compactification. Moreover, the suitable compactification often induces a scalar potential. Hence, here we consider the conditions for the coefficients which allow the de Sitter space solution. In other words, we assume the possibility of a more general effective action like in the previous section.

In the de Sitter space, the Hubble rate H is a constant

$$H = H_0 , \tag{56}$$

and all the curvatures are covariantly constant. We also assume the scalar field ϕ to be a constant:

$$\phi = p_0 . \tag{57}$$

For simplicity, c_3 terms are neglected and the scalar potential $V(\phi)$ is assumed to be given by

$$V(\phi) = V_0 \mathrm{e}^{-2\phi/\phi_0} . \tag{58}$$

Then the scalar equation has the following form:

$$0 = V_0 + 24 c_1 \alpha' x^2 + 96 c_2 \alpha'^2 x^3 , \quad x \equiv H_0 \mathrm{e}^{2p_0/\phi_0} . \tag{59}$$

On the other hand, the Friedmann equation is reduced to

$$0 = V_0 + 3x - 12 c_2 \alpha'^2 x^3 . \tag{60}$$

By eliminating c_2 from (59) and (60), one obtains

$$0 = 3V_0 + 8x + 8c_1 \alpha' x^2 . \tag{61}$$

On the other hand, by eliminating V_0 from (59) and (60), we get

$$0 = -1 + 8c_1 \alpha' x + 36 \alpha'^2 c_2 x^2 . \tag{62}$$

And by further eliminating the x^2 term from (61) and (62), we find

$$x = x_0 \equiv \frac{12 \alpha' c_2 V_0 + c_1}{8 \alpha' (c_1^2 - 4\alpha' c_2)} . \tag{63}$$

This expression for x is not always a solution of the two independent equations (59) and (60), or equivalently (61) and (62). The condition for x in (62) to be a solution can be obtained by substituting (63) into (61) (or equivalently (62)):

$$0 = 24 V_0 \alpha' (c_1^2 - 4\alpha' c_2)^2 + 8(12\alpha' c_2 V_0 + c_1)(c_1^2 - 4\alpha' c_2)$$
$$+ c_1 (12\alpha' c_2 V_0 + c_1)^2 . \tag{64}$$

In the particular case when $V_0 = 0$, (64) has the following form:

$$9c_1^2 = 32\alpha' c_2 , \tag{65}$$

and (63) gives

$$x_0 = \frac{c_1}{8\alpha' (c_1^2 - 4\alpha' c_2)} = -\frac{1}{\alpha' c_1} . \tag{66}$$

Therefore, if $c_1 < 0$, there is a possibility that there could occur a de Sitter space solution. We should note that, even if the solution exists, the Hubble rate H itself cannot be determined uniquely. In fact, since

$$H = \sqrt{x_0} \mathrm{e}^{p_0/\phi_0} , \tag{67}$$

by choosing p_0 properly, the value of H itself could be arbitrarily changed. Then the value of H could be determined by the initial condition.

Hence, we have presented in the above the condition to be satisfied by the coefficients of our effective action, which leads naturally to a de Sitter universe. In other words, if this condition is fulfilled, the early-time and (or) late-time universe can be inflationary (non-singular) due to stringy effects. It goes without saying that, again, the stability of such a de Sitter universe should be checked in each case, as was done for the bosonic string earlier.

7 Conclusions

In this paper we have considered string loop corrections to the Einstein–Hilbert action given by (3), with a dynamical dilaton ϕ. We have explored the cosmological dynamics of the corresponding modified gravity in the framework

of a low-energy string effective action. For simplicity, we have ignored the contribution of the background (radiation/matter) energy density. The higher-order string corrections to gravity, specially the third-order correction in α', crucially depend upon the string type. The evolution equations are quite involved and it is difficult to analyze them in general. Taking a different route, we have conjectured a particular solution, $H = h_0/t$, $\phi = \phi_0 \ln t/t_1$ for $h_0 > 0$ ($H = h_0/(t_s - t)$, $\phi = \phi_0 \ln(t_s - t)/t_1$ when $h_0 < 0$). This solution is important from the dark-energy viewpoint; we have carefully checked its viability by enforcing a consistency check on the parameters of the solution. This consistency requirement constrains the range of the parameter h_0, which defines the effective equation of state.

The model based upon a type II string turned out to be the simplest to investigate semi-analytically. The possible range of h_0 in this case is given by $0 < h_0 < 1/3$ corresponding to an uninteresting equation of state ($w_{\mathrm{eff}} > 1$). Type II is clearly ruled out because the string expansion parameter cannot be negative. Nevertheless, the situation might be improved in presence of matter. It would also be interesting to examine the fate of a phantom universe in presence of higher-curvature corrections with a massless dilaton.

In the case of the heterotic string, the dark energy solution exists for h_0 varying as $0.8 < h_0 < 5.04$, which corresponds to $w_{\mathrm{eff}} \geq -0.868$. WMAP$_3$ data analyzed with the SNLS survey constrains the dark energy equation of state as $w_{\mathrm{DE}} = -0.97^{+0.07}_{+0.09}$ at the 68% confidence level, which puts the heterotic string model under pressure. However, the combined (CMB + SNLS + LSS) data force the dark energy equation of state parameter to vary as $-1.001 < w_{\mathrm{DE}} < -0.875$. Thus, the heterotic string is only marginally compatible with observations.

Cosmological dynamics based upon the bosonic string turn out to be distinguished amongst all possible string types. Indeed, the consistency of the model leads to an effective equation of state given by $-1 \leq w_{\mathrm{eff}} \leq -0.17$, which is clearly compatible with the data. The stability analysis is difficult to carry out in this case, in general. We have studied the de Sitter case ($h_0 \to \infty$) in the bosonic case separately and demonstrated its stability.

A more general action, with higher-curvature corrections coefficients depending on a dilaton was also considered. The general reconstruction method could be developed for such theory, so that a realistic universe expansion history can be obtained within some class of scalar potentials. An example that proposes a matter-dominance era before cosmic acceleration (quintessence, phantom era or ΛCDM cosmology) is presented. The de Sitter universe in such a general theory (as well as for the bosonic string) can arise quite naturally. It is known that, with the addition of a scalar-GB term only, the low-energy string effective action can indeed help in the resolution of the initial singularity problem [30–37]. The appearance of de Sitter solution in the general case with higher curvature corrections clearly indicates that the resolution of an initial and/or a final singularity of any type (for the classification of future, finite-time singularities, see [38]) could be possible taking into account higher-order string loops.

We conclude that it is not so easy to get the natural dark energy universe from a low-energy string effective action (at the very least up to R^4 corrections). It could turn out that use of even higher order terms is necessary, or that the consideration of a different compactification might lead to a more realistic universe. In this respect, it could be expected that taking into account stringy non-perturbative effects might help. For instance, one future possibility is to consider not only higher-curvature corrections but also to include negative powers of such terms (an inverse α' expansion?) like in the models with positive and negative powers of the curvature [39].

Appendix: General evolution equations

The general evolution equations are obtained by varying the action (1) with respect to ϕ and the lapse function N. The scalar field equation takes the following form:

$$
\begin{aligned}
0 = {} & \ddot{\phi} + 3H\dot{\phi} \\
& + \frac{1}{\phi_0}\Big\{-48c_1\alpha' \mathrm{e}^{2\phi/\phi_0} H^4 - 192c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^6 \\
& + c_3[432\zeta(3) - 2736\delta_{\mathrm{B}} - 432\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^8 \\
& - 48c_1\alpha' \mathrm{e}^{2\phi/\phi_0} H^2 \dot{H} - 288c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^4 \dot{H} \\
& + c_3[864\zeta(3) - 5472\delta_{\mathrm{B}} - 864\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^6 \dot{H} \\
& - 288c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^2 \dot{H}^2 \\
& + c_3[792\zeta(3) - 7056\delta_{\mathrm{B}} + 1152\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^4 \dot{H}^2 \\
& - 96c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} \dot{H}^3 \\
& + c_3[288\zeta(3) - 4320\delta_{\mathrm{B}} - 720\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^2 \dot{H}^3 \\
& + c_3[36\zeta(3) - 1080\delta_{\mathrm{B}} - 180\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} \dot{H}^4\Big\}.
\end{aligned}
\tag{A.1}
$$

The Friedmann equation in the general case is given by

$$
\begin{aligned}
3H^2 = {} & \frac{1}{2}\dot{\phi}^2 + 24c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^6 \\
& - c_3[72\zeta(3) - 456\delta_{\mathrm{B}} - 72\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^8 \\
& - 432c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^4 \dot{H} \\
& + c_3[792\zeta(3) - 7056\delta_{\mathrm{B}} - 1152\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^6 \dot{H} \\
& - 288c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^2 \dot{H}^2 \\
& + c_3[828\zeta(3) - 10\,008\delta_{\mathrm{B}} - 1656\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^4 \dot{H}^2 \\
& + 48c_2\alpha'^2 \mathrm{e}^{4\phi/\phi_0} \dot{H}^3
\end{aligned}
$$

$$+ c_3[168\zeta(3) - 3600\delta_{\mathrm{B}} - 600\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^2 \dot{H}^3$$

$$- c_3[18\zeta(3) - 540\delta_{\mathrm{B}} - 90\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} \dot{H}^4$$

$$- 144 c_2 \alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^3 \ddot{H}^3$$

$$+ c_3[264\zeta(3) - 2352\delta_{\mathrm{B}} - 384\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^5 \ddot{H}$$

$$- 144 c_2 \alpha'^2 \mathrm{e}^{4\phi/\phi_0} H \dot{H} \ddot{H}$$

$$+ c_3[288\zeta(3) - 4320\delta_{\mathrm{B}} - 720\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^3 \dot{H} \ddot{H}$$

$$+ c_3[72\zeta(3) - 2160\delta_{\mathrm{B}} - 360\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H \dot{H}^2 \ddot{H}$$

$$+ \frac{\dot{\phi}}{\phi_0}\Big\{ -48 c_1 \alpha' \mathrm{e}^{2\phi/\phi_0} H^3 - 288 c_2 \alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^5$$

$$+ c_3[864\zeta(3) - 5472\delta_{\mathrm{B}} - 864\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^7$$

$$- 576 c_2 \alpha'^2 \mathrm{e}^{4\phi/\phi_0} H^3 \dot{H}$$

$$+ c_3[1584\zeta(3) - 14\,112\delta_{\mathrm{B}} - 2304\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^5 \dot{H}$$

$$- 288 c_2 \alpha'^2 \mathrm{e}^{4\phi/\phi_0} H \dot{H}^2$$

$$+ c_3[864\zeta(3) - 12\,960\delta_{\mathrm{B}} - 2160\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H^3 \dot{H}^2$$

$$+ c_3[144\zeta(3) - 4320\delta_{\mathrm{B}} - 720\delta_{\mathrm{H}}]\alpha'^3 \mathrm{e}^{6\phi/\phi_0} H \dot{H}^3 \Big\}.$$
$$(\mathrm{A.2})$$

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